

Initial problem for heat equation with multisoliton inhomogeneity and one-loop quantum corrections

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Abstract

The generalized zeta-function is built by a dressing method based on the Darboux covariance of the heat equation and used to evaluate the correspondent functional integral in quasiclassical approximation. Quantum corrections to a kink-like solutions of Landau-Ginzburg model are calculated.

1 Introduction

In the paper of V.Konoplich [1] quantum corrections to a few classical solutions by means of Riemann zeta-function are calculated. Most interesting of them are the corrections to the kink - the separatrix solution of field ϕ^4 model. The method of [1] is rather complicated and it could be useful to simplify it. We use the dressing technique based on classical Darboux transformations (DT) with a new applications to Green function construction [2]. It is the main aim of this note with eventual possibility to generalize the result due to universality of the technique when a link to integrable (soliton, SUSY)

systems is established [3]. The suggested approach open new possibilities; for example it allows to show the way to calculate the quantum corrections to Q-balls [4] and periodic solutions of the models. The last problem is posed in the useful review [5].

2 Heat equation Cauchy problem

We will base on the DT-covariance of the heat equation for the function $\rho(\tau, x, y)$

$$-\rho_\tau + \rho_{xx} + u(x)\rho = 0, \quad (1)$$

that means the form-invariance of (1) with respect to iterated DT, defined by the Wronskian $W[\phi_1, \dots, \phi_N]$ of the solutions of (1)

$$\begin{aligned} \rho &\rightarrow \rho[N] = \frac{W[\phi_1, \dots, \phi_N, \rho]}{W[\phi_1, \dots, \phi_N]}, \\ u &\rightarrow u[N] = u + 2\ln_{xx} W[\phi_1, \dots, \phi_N]. \end{aligned} \quad (2)$$

Consider now a Cauchy problem for the equation (1), where $u(x)$ represents the reflectionless potential [7] in a sense that it could be produced by the DT and the initial condition is

$$\rho(0, x, y) = \delta(x - y). \quad (3)$$

The problem is formulated for a Green function: it is rather general and may be applied as a model of classical diffusion or heat conductivity. We, however, would follow other applications in the theory of quasiclassical quantization, where the function ρ is treated as density matrix whence τ stands for inverse temperature [?].

The algorithm of such problem solution is the dressing procedure organized by a sequence of DTs defined by (2):

$$\begin{aligned} (\frac{\partial}{\partial x} - \ln_x \phi_1(x, y))\rho_0(0, x, y) &= g_1(x, y), \\ (\frac{\partial}{\partial x} - \ln_x \phi_2[1](x, y))g_1(x, y) &= g_2(x, y), \dots, \\ (\frac{\partial}{\partial x} - \ln_x \phi_k[k-1](x, y))g_{k-1} &= g_k(x, y), \\ g_N(x, y) &= \delta(x, y), 2 \leq k \leq N. \end{aligned} \quad (4)$$

and the following theorem

Theorem The function $\rho[N]$ being built by (2) will be a solution of the problem (1,3) with the potential $u[N]$, if $\rho(\tau, x, y)$ is a solution of the (1) with the initial condition $\rho_0(0, x, y)$.

The result is used when static solutions of ϕ^4 model are quantized by means of Riemann function $\zeta(s)$ [1] expressed via the Green functions of the

(1) (see also [2]). The one-loop quantum correction to action is evaluated directly as

$$S_q = -\zeta'(0).$$

3 Example of kink

Most popular example of the kink is obtained in this scheme by means of DT over zero seed $u = 0$. The solution ρ of (1) with ρ_0 as initial condition for this case is a simple heat equation solution

$$\rho(\tau, x, y) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \rho_0(z, y) \exp[-(x - z)^2/4\tau] dz. \quad (5)$$

The initial condition ρ_0 is evaluated by direct integration in (4):

$$\rho_0(x, y) = \phi_1(x) \begin{cases} \phi_1^{-1}(y), & x > y \\ 0, & x < y \end{cases} \quad (6)$$

The Green function $\rho[2]$ (density matrix) for the kink solution as the potential is built by the two-fold DT by the Wronskian formula (2) that results in

$$\begin{aligned} \rho[2](\tau, x, y) = & \exp\left[\frac{-(x-y)^2}{4\tau}\right] / 2\sqrt{\pi\tau} + \\ & + \frac{1}{2} \sum_{m=1}^2 \rho_m \psi_m(x) \psi_m(y) [Erf\left[\frac{(x-y+2b_m\tau)}{2\sqrt{\tau}}\right] - Erf\left[\frac{(x-y-2b_m\tau)}{2\sqrt{\tau}}\right]], \end{aligned} \quad (7)$$

where $b_k = km/\sqrt{2}$, $\rho_k = ||\psi||^{-2}$, $k=1,2$. After multiplication of the Green function by $\exp[-4m^2\tau]$:

$$\rho \rightarrow \rho \exp[-4m^2\tau],$$

the first term of the Green function leads to a divergent integral. This divergency is well-known, its origin is a zero vacuum oscillations. In our approach this fact has transparent explanation, because the divergent term is simply a solution of heat equation with constant coefficients, that appear when the self-action of scalar field is neglected. Such divergence is usually compensated by addition of contra terms of normal order.

Our procedure deletes all ultraviolet divergencies of 1+1 ϕ^4 model including energy of zero oscillations and one-meson states if one evaluates the generalized zeta-function by the formula

$$\zeta_D(s) = M^{2s} \frac{\int_0^\infty \gamma(t) t^{s-1} dt}{\Gamma(s)} \quad (8)$$

$\Gamma(s)$ is the Euler gamma function and M is a mass scale. The function $\gamma(t)$ in the integrand of (8) is expressed via the Green functions $G(x, y, \tau)$ and $G_0(x, y, \tau)$ difference. The result coincides with one from [1].

4 Conclusion

As a conclusion let us note that this approach is elaborated in [6] (published in a local conference abstract book) and allows to calculate one-loop corrections to the N-level reflectionless potential and, very similarly, solitons of SG. Some eventual applications are visible in the case studied at [8].

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